

**An overview of beamformer approaches:**

**Basic principles, underlying assumptions,  
and major causes of errors**

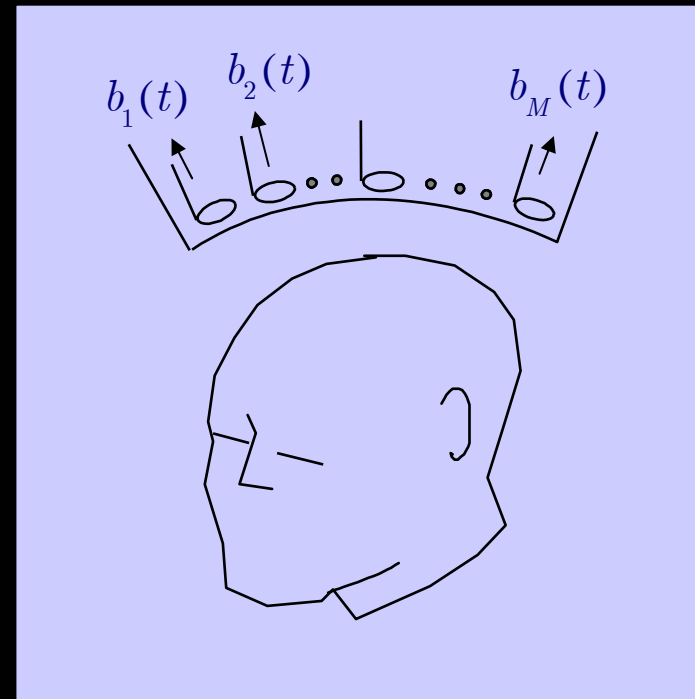
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## Definitions

- array measurements:  $\mathbf{b}(t) =$

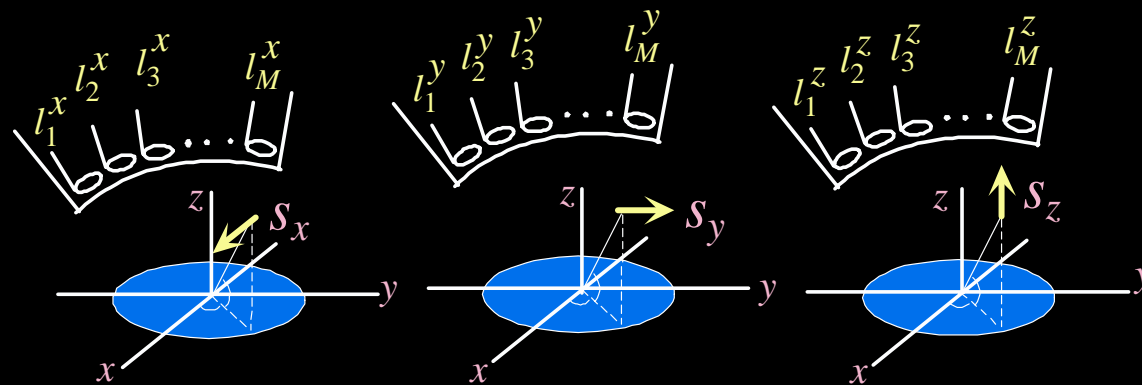
$$\begin{bmatrix} b_1(t) \\ b_2(t) \\ \vdots \\ b_M(t) \end{bmatrix}$$



- data covariance matrix:  $\mathbf{R} = \langle \mathbf{b}(t)\mathbf{b}^T(t) \rangle$
- source magnitude:  $s(\mathbf{r}, t)$
- source orientation:  $\boldsymbol{\eta}(\mathbf{r}, t) = [\eta_x(\mathbf{r}, t), \eta_y(\mathbf{r}, t), \eta_z(\mathbf{r}, t)]$

# Sensor lead field

$$\mathbf{L}(\mathbf{r}) = \begin{bmatrix} l_1^x(\mathbf{r}) & l_1^y(\mathbf{r}) & l_1^z(\mathbf{r}) \\ l_2^x(\mathbf{r}) & l_2^y(\mathbf{r}) & l_2^z(\mathbf{r}) \\ \vdots & \vdots & \vdots \\ l_M^x(\mathbf{r}) & l_M^y(\mathbf{r}) & l_M^z(\mathbf{r}) \end{bmatrix}, \quad \mathbf{l}(\mathbf{r}) = \mathbf{L}(\mathbf{r}) \begin{bmatrix} \eta_x(\mathbf{r}) \\ \eta_y(\mathbf{r}) \\ \eta_z(\mathbf{r}) \end{bmatrix}$$



$$|s_x| = |s_y| = |s_z| = 1$$

## Spatial filter

$$\hat{s}(\mathbf{r}, t) = \mathbf{w}^T(\mathbf{r})\mathbf{b}(t) = [w_1(\mathbf{r}), \dots, w_M(\mathbf{r})] \begin{bmatrix} b_1(t) \\ \vdots \\ b_M(t) \end{bmatrix} = \sum_{m=1}^M w_m(\mathbf{r})b_m(t)$$

↑  
weight vector

## Minimum-variance spatial filter

$$\min_{\mathbf{w}} \mathbf{w}^T \mathbf{R} \mathbf{w} \text{ subject to } \mathbf{w}^T \mathbf{l}(\mathbf{r}) = 1 \Rightarrow \mathbf{w}^T(\mathbf{r}) = \frac{\mathbf{l}^T(\mathbf{r})\mathbf{R}^{-1}}{\mathbf{l}^T(\mathbf{r})\mathbf{R}^{-1}\mathbf{l}(\mathbf{r})}$$

Assume  $Q$  sources are located at  $\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_Q$

Outputs of spatial filter pointing at  $\mathbf{r}_p$  (with  $\mathbf{w}^T(\mathbf{r}_p)\mathbf{l}(\mathbf{r}_p) = 1$ )

$$\begin{aligned} &\Downarrow \\ \min \mathbf{w}^T(\mathbf{r}_p)\mathbf{R}\mathbf{w}(\mathbf{r}_p) &= \langle s(\mathbf{r}_p, t)^2 \rangle + \sum_{q \neq p} \langle s(\mathbf{r}_q, t)^2 \rangle \left\| \mathbf{w}^T(\mathbf{r}_p)\mathbf{l}(\mathbf{r}_q) \right\| \\ &\quad \uparrow \qquad \qquad \qquad \Downarrow \\ &\quad \langle s(\mathbf{r}_p, t)s(\mathbf{r}_q, t) \rangle = 0 \qquad \text{zero} \end{aligned}$$

Therefore, this minimization gives the weight satisfying

$$\begin{aligned} \mathbf{w}^T(\mathbf{r}_p)\mathbf{l}(\mathbf{r}_q) &= 1 \quad \text{for } p = q \\ &= 0 \quad \text{for } p \neq q \end{aligned}$$

Vital assumption:  $\langle s(\mathbf{r}_p, t)s(\mathbf{r}_q, t) \rangle = 0$  for  $p \neq q$

## Low-rank signal assumption

Consider a simplest case where we know locations and orientations of all  $Q$  sources

weight  $\mathbf{w}(\mathbf{r}_1)$  (containing  $M$  unknowns) can be obtained by solving a set of  $Q$  linear equations:

$$\mathbf{w}^T(\mathbf{r}_1)\mathbf{l}(\mathbf{r}_1) = w_1(\mathbf{r}_1)l_1(\mathbf{r}_1) + \dots + w_M(\mathbf{r}_1)l_M(\mathbf{r}_1) = 1$$

$$\mathbf{w}^T(\mathbf{r}_1)\mathbf{l}(\mathbf{r}_2) = w_1(\mathbf{r}_1)l_1(\mathbf{r}_2) + \dots + w_M(\mathbf{r}_1)l_M(\mathbf{r}_2) = 0$$

⋮

$$\mathbf{w}^T(\mathbf{r}_1)\mathbf{l}(\mathbf{r}_Q) = w_1(\mathbf{r}_1)l_1(\mathbf{r}_Q) + \dots + w_M(\mathbf{r}_1)l_M(\mathbf{r}_Q) = 0$$

If  $Q > M$ , there is no solution for  $\mathbf{w}^T(\mathbf{r}_1)$

$$Q < M$$

## Fundamental assumptions for adaptive beamformer source reconstruction:

- Sources are uncorrelated.
- Signals are low rank.

## Vector source detection

The electromagnetic sources are three dimensional vectors.



The minimum-variance beamformer should incorporate the vector nature of sources.

Two-types of formulations have been proposed:

- (1) Scalar formulation
- (2) Vector formulation



## Scalar formulation

$$\hat{s}(\mathbf{r}, t) = \mathbf{w}^T(\mathbf{r}, \boldsymbol{\eta})\mathbf{b}(t) \approx \mathbf{w}^T(\mathbf{r}, \boldsymbol{\eta}_{opt})\mathbf{b}(t)$$

$\uparrow$

$$\boldsymbol{\eta}_{opt} = \arg \max_{\boldsymbol{\eta}} \langle \hat{s}(\mathbf{r}, t)^2 \rangle = \arg \max_{\boldsymbol{\eta}} \mathbf{w}^T(\mathbf{r}, \boldsymbol{\eta})\mathbf{R}\mathbf{w}(\mathbf{r}, \boldsymbol{\eta})$$

## Scalar MV beamformer

$$\boldsymbol{\eta}_{opt} = \arg \max_{\boldsymbol{\eta}} \langle \hat{s}(\mathbf{r}, t)^2 \rangle = \left[ \arg \min_{\boldsymbol{\eta}} (\boldsymbol{\eta}^T \mathbf{L}^T(\mathbf{r})\mathbf{R}^{-1}\mathbf{L}(\mathbf{r})\boldsymbol{\eta}) \right]^{-1} = \mathbf{u}_{min}$$

$\uparrow$

eigenvector corresponding to the minimum eigenvalue of  $[\mathbf{L}^T(\mathbf{r})\mathbf{R}^{-1}\mathbf{L}(\mathbf{r})]$

$$\mathbf{w}^T(\mathbf{r}, \boldsymbol{\eta}) = \frac{\mathbf{u}_{min}^T \mathbf{L}^T(\mathbf{r})\mathbf{R}^{-1}}{\mathbf{u}_{min}^T \mathbf{L}^T(\mathbf{r})\mathbf{R}^{-1}\mathbf{L}(\mathbf{r})\mathbf{u}_{min}}$$

## Vector formulation

uses three weight vectors which detect  $x$ ,  $y$ , and  $z$  source components.

$$[\hat{s}_x(\mathbf{r}), \hat{s}_y(\mathbf{r}), \hat{s}_z(\mathbf{r})] = [\mathbf{w}_x(\mathbf{r}), \mathbf{w}_y(\mathbf{r}), \mathbf{w}_z(\mathbf{r})]^T \mathbf{b}(t)$$

## Vector MV beamformer formulation

$$\min_{\mathbf{w}_x} \mathbf{w}_x^T \mathbf{R} \mathbf{w}_x \text{ subject to } \mathbf{w}_x^T \mathbf{l}_x = 1, \mathbf{w}_x^T \mathbf{l}_y = 0, \mathbf{w}_x^T \mathbf{l}_z = 0$$

$$\min_{\mathbf{w}_y} \mathbf{w}_y^T \mathbf{R} \mathbf{w}_y \text{ subject to } \mathbf{w}_y^T \mathbf{l}_x = 0, \mathbf{w}_y^T \mathbf{l}_y = 1, \mathbf{w}_y^T \mathbf{l}_z = 0$$

$$\min_{\mathbf{w}_z} \mathbf{w}_z^T \mathbf{R} \mathbf{w}_z \text{ subject to } \mathbf{w}_z^T \mathbf{l}_x = 0, \mathbf{w}_z^T \mathbf{l}_y = 0, \mathbf{w}_z^T \mathbf{l}_z = 1$$



$$[\mathbf{w}_x(\mathbf{r}), \mathbf{w}_y(\mathbf{r}), \mathbf{w}_z(\mathbf{r})]^T = [\mathbf{L}^T(\mathbf{r}) \mathbf{R}^{-1} \mathbf{L}(\mathbf{r})]^{-1} \mathbf{L}^T(\mathbf{r}) \mathbf{R}^{-1}$$

# Equivalence between the two formulations -output power

Scalar formulation:

$$\max_{\boldsymbol{\eta}} \langle \hat{s}(\mathbf{r}, t)^2 \rangle = \max_{\boldsymbol{\eta}} \frac{1}{\boldsymbol{\eta}^T \mathbf{L}^T(\mathbf{r}) \mathbf{R}^{-1} \mathbf{L}(\mathbf{r}) \boldsymbol{\eta}} = \frac{1}{\gamma_{min}}$$

Vector formulation:

$$\max_{\boldsymbol{\eta}} \langle \hat{s}(\mathbf{r}, t)^2 \rangle = \max_{\boldsymbol{\eta}} \left\| [\hat{s}_x(\mathbf{r}), \hat{s}_y(\mathbf{r}), \hat{s}_z(\mathbf{r})] \boldsymbol{\eta} \right\|^2 = \frac{1}{\gamma_{min}}$$

$\gamma_{min}$  : minimum eigenvalue of  $[\mathbf{L}^T(\mathbf{r}) \mathbf{R}^{-1} \mathbf{L}(\mathbf{r})]$

Two formulations give the same output power if the beamformer is set at the optimum orientation.

# Equivalence between the two formulations -output SNR

## Scalar beamformer

$$Z_S^{opt}(\mathbf{r}) = \max_{\boldsymbol{\eta}} \frac{\langle \hat{s}(\mathbf{r}, \boldsymbol{\eta}, t)^2 \rangle}{\sigma_0^2 \|\mathbf{w}(\mathbf{r}, \boldsymbol{\eta})\|^2} = \frac{1}{\sigma_0^2 \alpha_{min}}$$

## Vector beamformer

$$Z_V^{opt}(\mathbf{r}) = \max_{\boldsymbol{\eta}} \frac{\|[\hat{s}_x(\mathbf{r}), \hat{s}_y(\mathbf{r}), \hat{s}_z(\mathbf{r})]\boldsymbol{\eta}\|^2}{\sigma_0^2 \|\mathbf{W}(\mathbf{r})\boldsymbol{\eta}\|^2} = \frac{1}{\sigma_0^2 \alpha_{min}}$$

$\alpha_{min}$  : minimum eigenvalue of  $[\mathbf{L}^T(\mathbf{r})\mathbf{R}^{-1}\mathbf{L}(\mathbf{r})]^{-1} [\mathbf{L}^T(\mathbf{r})\mathbf{R}^{-2}\mathbf{L}(\mathbf{r})]$

Two formulations give the same asymptotic output SNR if the beamformer is set at the optimum orientation.

## Bias of estimated source locations

Remarkable property of the adaptive beamformer is that the methods do not have a source location bias even in the presence of noise.

This is in contrast to the minimum-norm-based reconstruction methods where the source location bias more or less exists.

## Location bias for a single source

A single source at  $\mathbf{r}_1$

source lead field:  $\mathbf{f}$  , source power:  $\sigma_1^2$  , noise power:  $\sigma_0^2$

Condition for no location bias:

$$\underbrace{\sigma_1^2 \|\mathbf{w}^T(\mathbf{r}_1)\mathbf{f}\|^2 + \sigma_0^2 \|\mathbf{w}(\mathbf{r}_1)\|^2}_{\text{total power at source location}} > \underbrace{\sigma_1^2 \|\mathbf{w}^T(\mathbf{r})\mathbf{f}\|^2 + \sigma_0^2 \|\mathbf{w}(\mathbf{r})\|^2}_{\text{total power at any other locations}}$$

## Adaptive spatial filters

### Minimum variance

$$\mathbf{w}(\mathbf{r}) = \mathbf{R}^{-1} \mathbf{l}(\mathbf{r}) / [\mathbf{l}^T(\mathbf{r}) \mathbf{R}^{-1} \mathbf{l}(\mathbf{r})] \Leftarrow \min_{\mathbf{w}} \mathbf{w}^T \mathbf{R} \mathbf{w} \text{ subject to } \mathbf{w}^T \mathbf{l}(\mathbf{r}) = 1$$

### Minimum variance with normalized lead field

$$\mathbf{w}(\mathbf{r}) = \mathbf{R}^{-1} \tilde{\mathbf{l}}(\mathbf{r}) / [\tilde{\mathbf{l}}^T(\mathbf{r}) \mathbf{R}^{-1} \tilde{\mathbf{l}}(\mathbf{r})] \Leftarrow \min_{\mathbf{w}} \mathbf{w}^T \mathbf{R} \mathbf{w} \text{ subject to } \mathbf{w}^T \mathbf{l}(\mathbf{r}) = |\mathbf{l}(\mathbf{r})|$$

### Weight-normalized minimum variance (Borgiotti-Kaplan)

$$\mathbf{w}(\mathbf{r}) = \mathbf{R}^{-1} \mathbf{l}(\mathbf{r}) / \sqrt{\mathbf{l}^T(\mathbf{r}) \mathbf{R}^{-2} \mathbf{l}(\mathbf{r})} \Leftarrow \min_{\mathbf{w}} \mathbf{w}^T \mathbf{R} \mathbf{w} \text{ subject to } \mathbf{w}^T \mathbf{w} = 1$$

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$$\tilde{\mathbf{l}}(\mathbf{r}) = \mathbf{l}(\mathbf{r}) / \|\mathbf{l}(\mathbf{r})\|$$

## Condition for no location bias

$$\sigma_1^2 \|\mathbf{w}^T(\mathbf{r}_1)\mathbf{f}\|^2 + \sigma_0^2 \|\mathbf{w}(\mathbf{r}_1)\|^2 > \sigma_1^2 \|\mathbf{w}^T(\mathbf{r})\mathbf{f}\|^2 + \sigma_0^2 \|\mathbf{w}(\mathbf{r})\|^2$$

Minimum-variance

$$\frac{\|\mathbf{f}\|}{\|\mathbf{l}\|} \frac{1}{1 + \alpha[1 - \cos^2(\mathbf{l}, \mathbf{f})]} < 1$$

Minimum variance with normalized lead field

$$\frac{1}{1 + \alpha[1 - \cos^2(\mathbf{l}, \mathbf{f})]} < 1$$

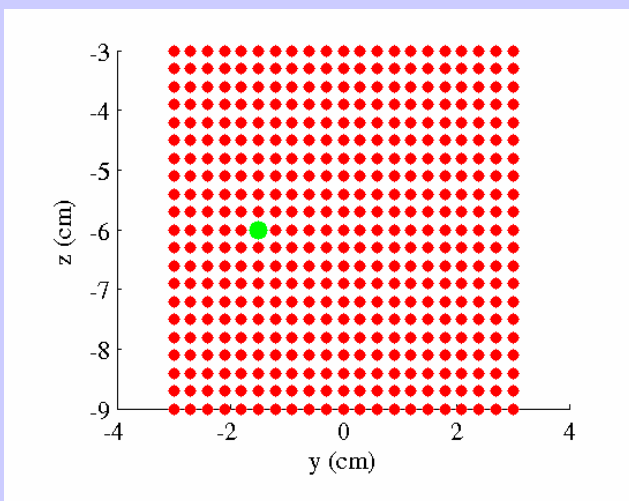
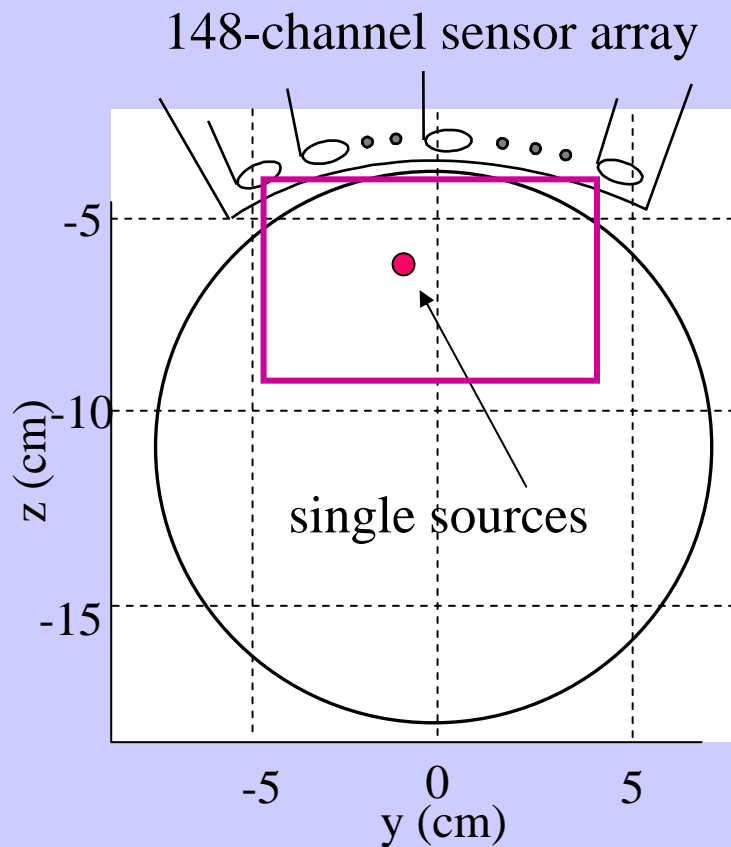
Weight-normalized minimum variance

$$\frac{1 + \alpha[1 - \cos^2(\mathbf{l}, \mathbf{f})]}{1 + \alpha[1 - \cos^2(\mathbf{l}, \mathbf{f})] + \alpha(\alpha + 1)[1 - \cos^2(\mathbf{l}, \mathbf{f})]} < 1$$

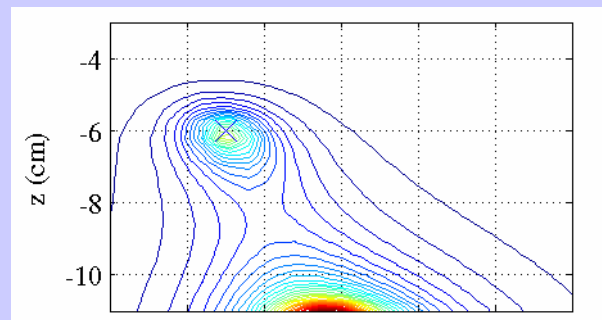
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$$\alpha = (\sigma_1^2 / \sigma_0^2) \|\mathbf{f}\|^2 \geq M, \quad 0 \leq \cos(\mathbf{l}, \mathbf{f}) = \frac{|\mathbf{l}^T \mathbf{f}|}{\|\mathbf{f}\| \|\mathbf{l}\|} \leq 1$$

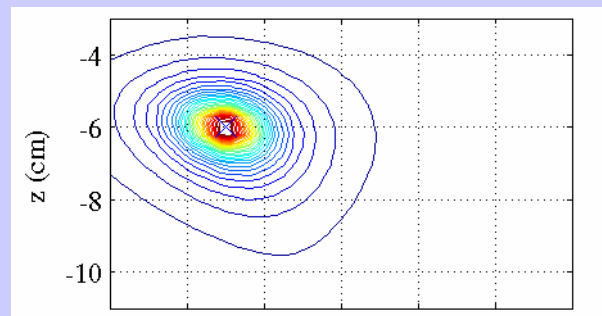




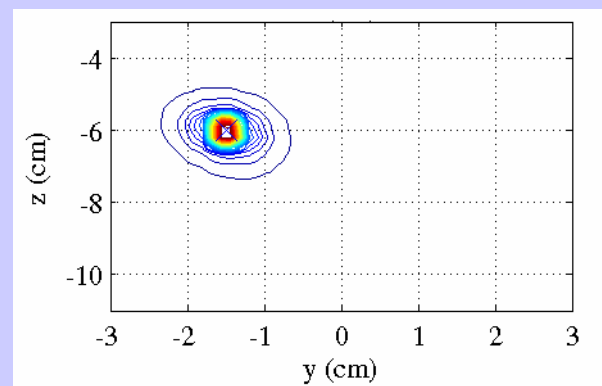
$$\alpha = M(148)$$



Minimum variance



Minimum variance (normalized lead field)



Weight-normalized minimum variance

(The cross mark indicates the source location.)

# Effect of noise on location bias

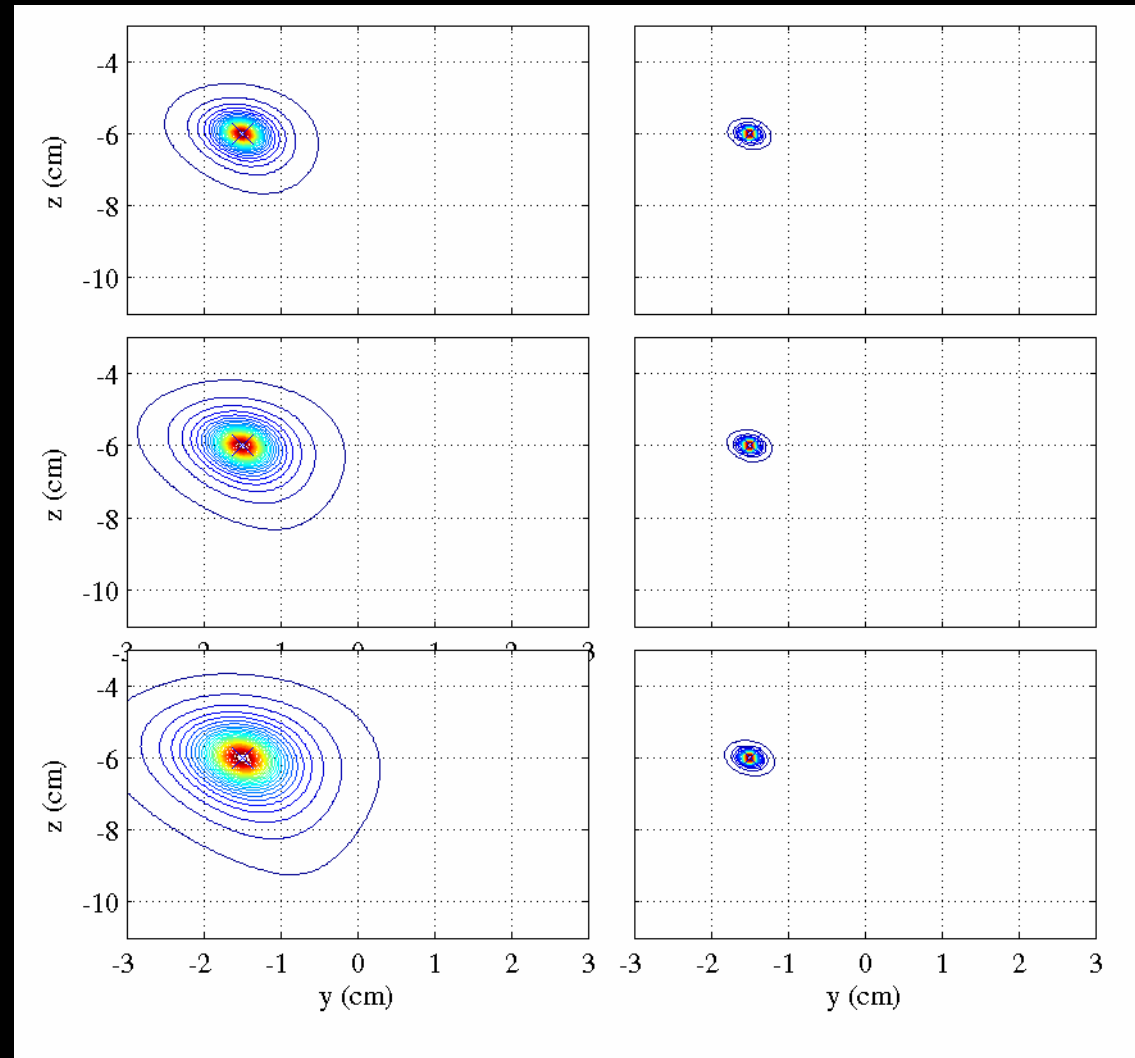
Minimum-variance (normalized lead field)      Weight-normalized minimum-variance

Input SNR

$$\alpha = 4M(592)$$

$$\alpha = 2M(296)$$

$$\alpha = 1M(148)$$



## Spatial resolution comparison

Reconstruction profile of a single source:  $\phi(\mathbf{r}) = \|\mathbf{w}^T(\mathbf{r})\mathbf{f}\| / \|\mathbf{w}^T(\mathbf{r}_1)\mathbf{f}\|$   
(Point-spread function)

Minimum variance:

$$\phi(\mathbf{r}) = \frac{\cos(l, \mathbf{f})}{1 + \alpha[1 - \cos^2(l, \mathbf{f})]}$$

Weight-normalized  
minimum variance:

$$\phi(\mathbf{r}) \approx \frac{1}{1 + \alpha^2[1 - \cos^2(l, \mathbf{f})]}$$

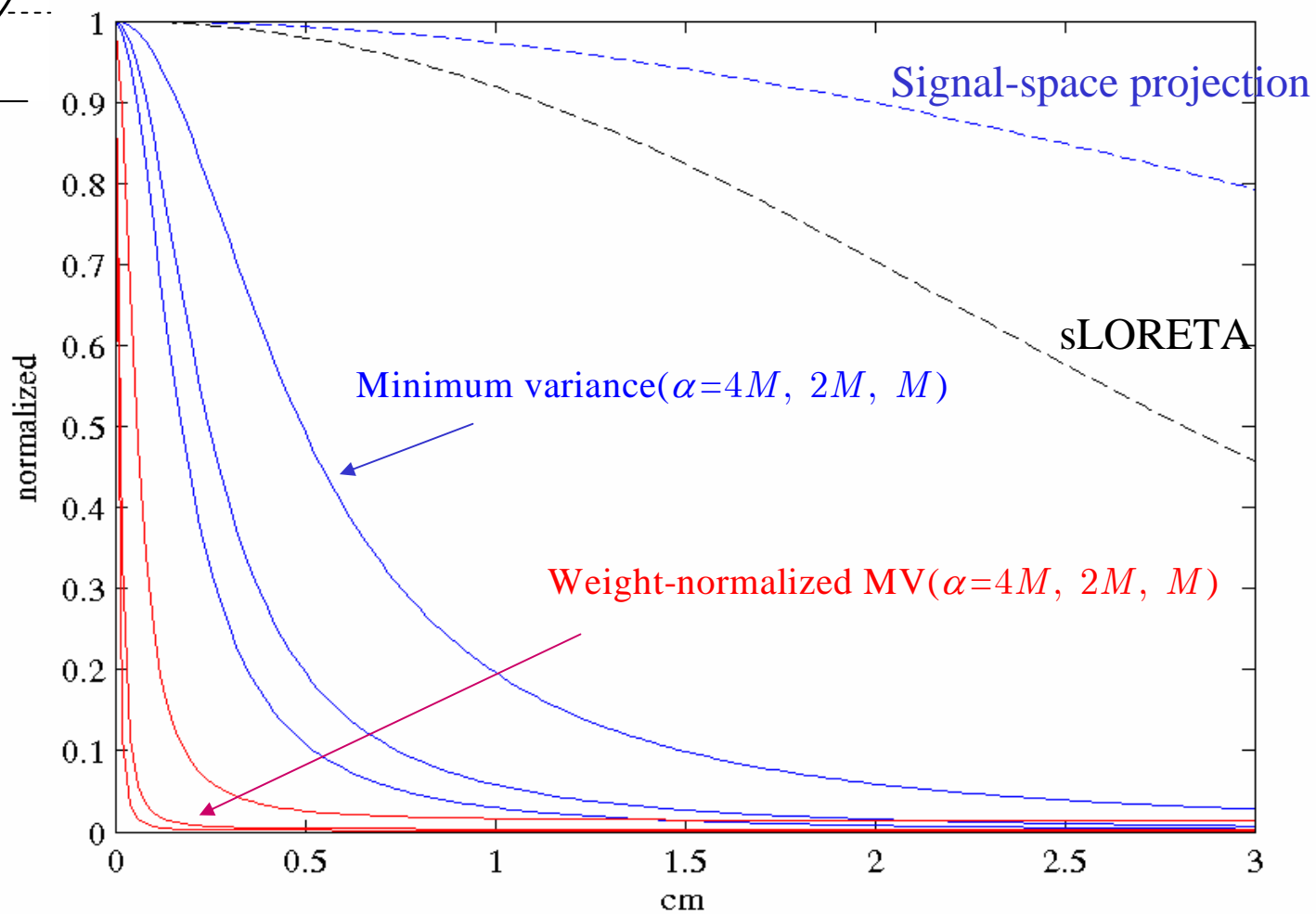
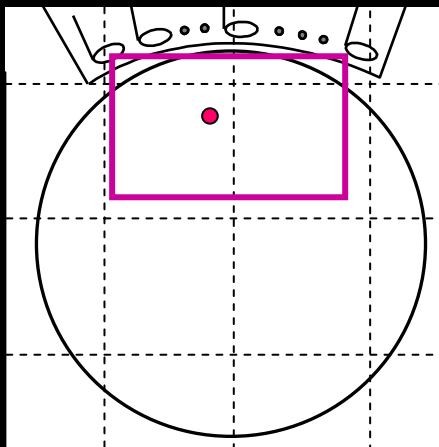
Signal-space projection:

$$\phi(\mathbf{r}) = \cos(l, \mathbf{f})$$

sLORETA:

$$\phi(\mathbf{r}) = \cos(l, \mathbf{f} | \mathbf{G}^{-1})$$

Because input SNR  $\alpha > M$ , these parts causes a rapid decay.



What happens if the assumptions that

- sources are uncorrelated,

- signals are low rank,

are not satisfied.

## Source correlation influence for adaptive spatial filters

$$\mathbf{w}^T(\mathbf{r}_p)\mathbf{l}(\mathbf{r}_q) = \delta_{pq} \quad (\text{Sources are uncorrelated})$$

$$\mathbf{w}^T(\mathbf{r}_p)\mathbf{l}(\mathbf{r}_q) = \frac{[\mathbf{R}_S^{-1}]_{pq}}{[\mathbf{R}_S^{-1}]_{pp}} \quad (\text{Sources are partially correlated})$$

When  $Q$  sources are correlated with the  $p$ th source,

$$\hat{s}(\mathbf{r}_p, t) = s(\mathbf{r}_p, t) + \sum_{q=1}^Q \frac{[\mathbf{R}_S^{-1}]_{pq}}{[\mathbf{R}_S^{-1}]_{pp}} s(\mathbf{r}_q, t)$$

↑  
spatial-filter output

↑  
leakages from other correlated sources

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$\mathbf{R}_S$  : source covariance matrix,  $[\mathbf{R}_S^{-1}]_{pq}$  : the  $(p, q)$  element of  $\mathbf{R}_S^{-1}$

# Signal cancellation

When two correlated sources exist

$$\hat{s}(\mathbf{r}_1, t) = s(\mathbf{r}_1, t) - \left(\frac{\alpha_1 \mu}{\alpha_2}\right) s(\mathbf{r}_2, t)$$

$$\hat{s}(\mathbf{r}_2, t) = -\left(\frac{\alpha_2 \mu}{\alpha_1}\right) s(\mathbf{r}_1, t) + s(\mathbf{r}_2, t)$$

⇓

$$\langle \hat{s}(\mathbf{r}_1, t)^2 \rangle = (1 - \mu^2) \langle s(\mathbf{r}_1, t)^2 \rangle$$

$$\langle \hat{s}(\mathbf{r}_2, t)^2 \rangle = (1 - \mu^2) \langle s(\mathbf{r}_2, t)^2 \rangle$$

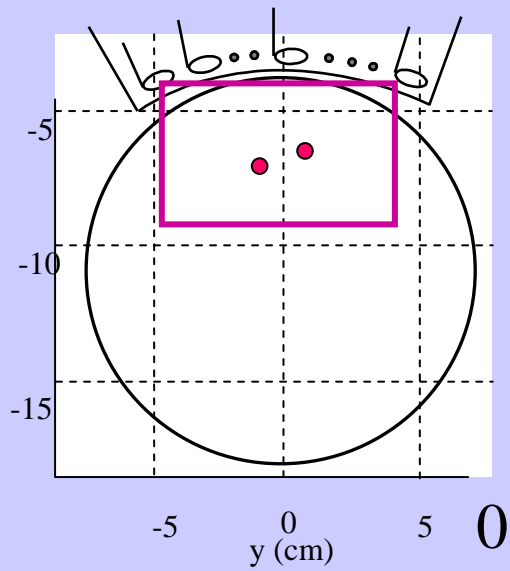
⇑

Source power decreases by a factor of  $(1 - \mu^2)$

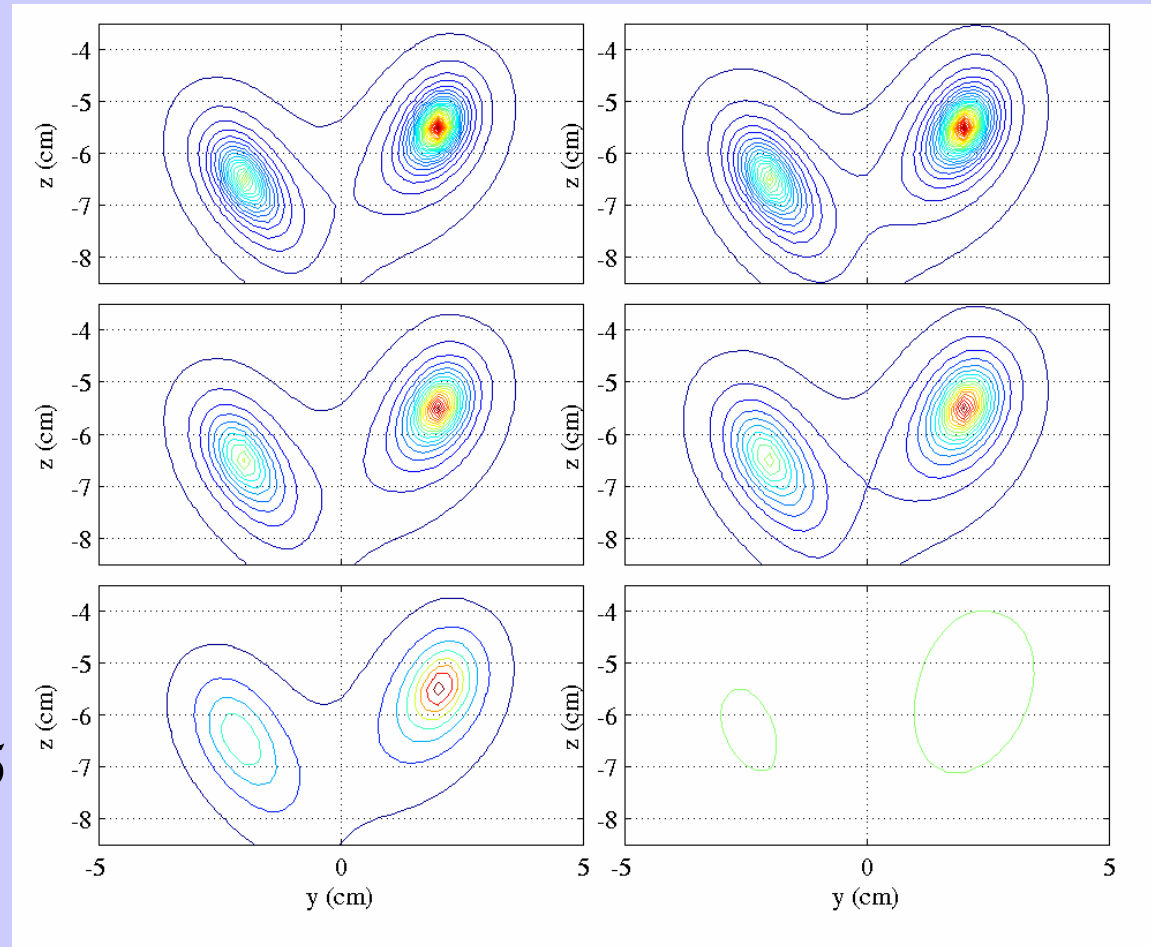
↑

source correlation coefficient

# Reconstruction experiments when correlated sources exist



0  
0.6  
0.85



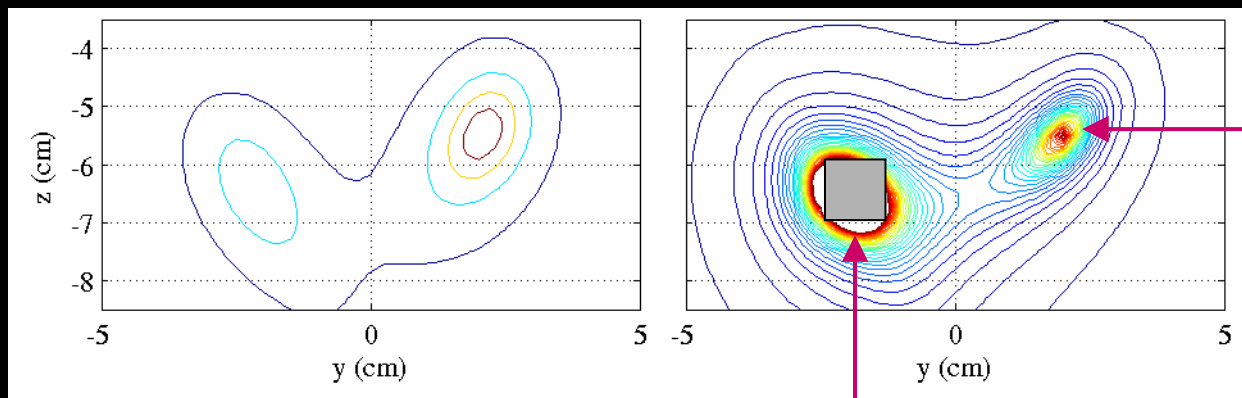
0.4  
0.7  
0.98



## Coherent source suppression:

If locations of coherent interferences are approximately known, its influence can be suppressed.

Correlation coefficient: 0.92



Recover the  
signal source of  
interest

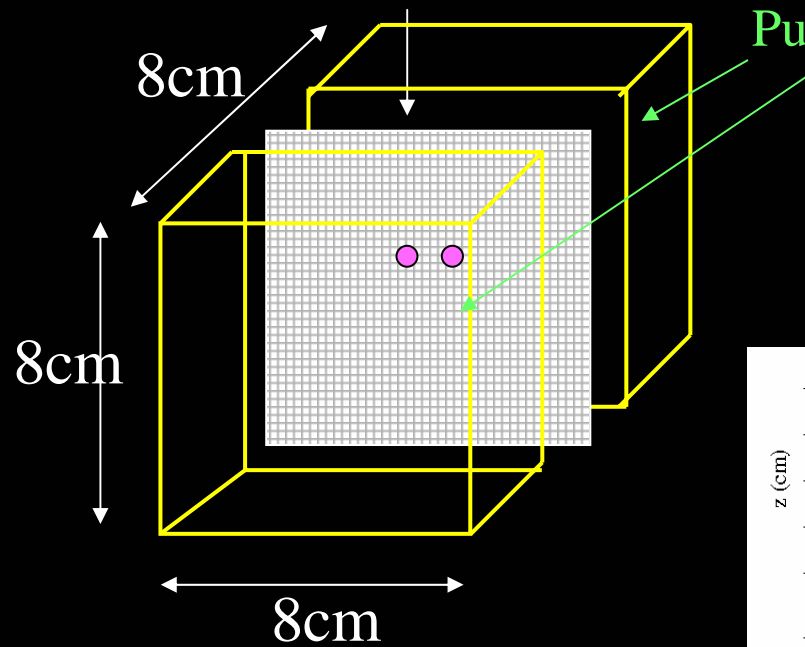
Impose the null sensitivity

The detail of this method is described in poster A153:

Sarang et al. "Modified Beamformers for Coherent Source Region Suppression"

# Effects of large number of noise sources

reconstruction plane



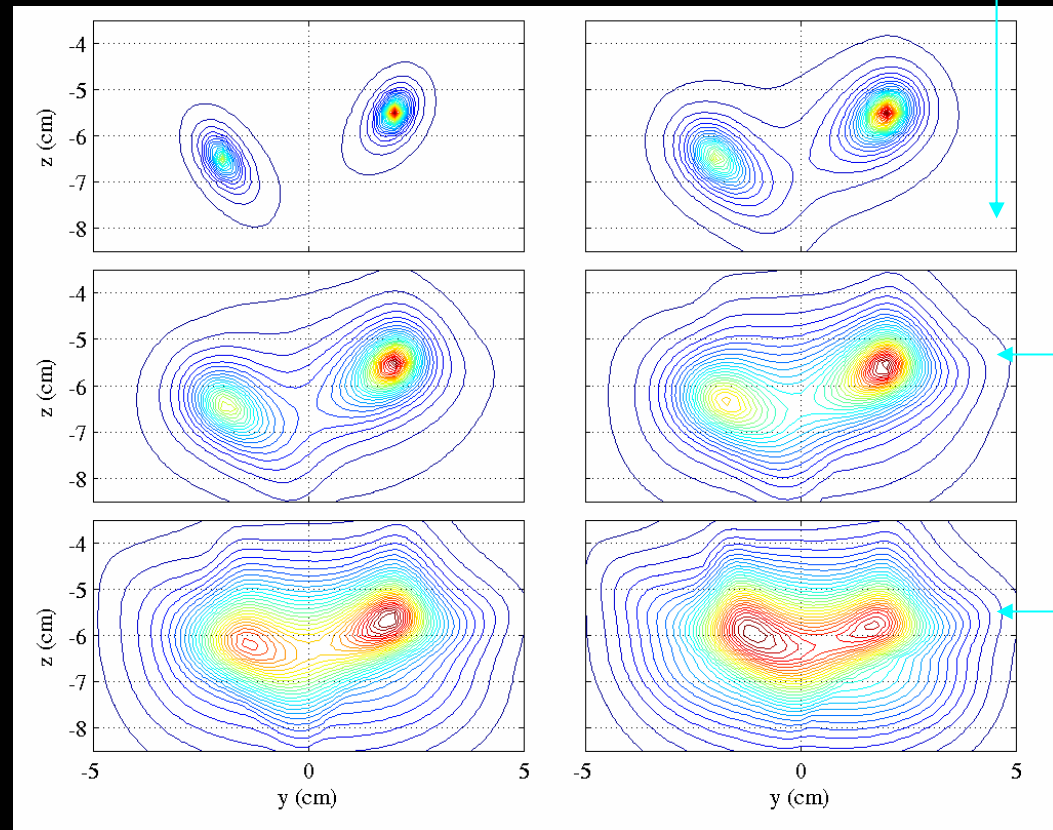
Put 100 noise sources in these regions

$$P_N = 0.1\sigma_2^2$$

$$P_N = 0.01\sigma_2^2$$

$$P_N = 0.001\sigma_2^2$$

No noise source

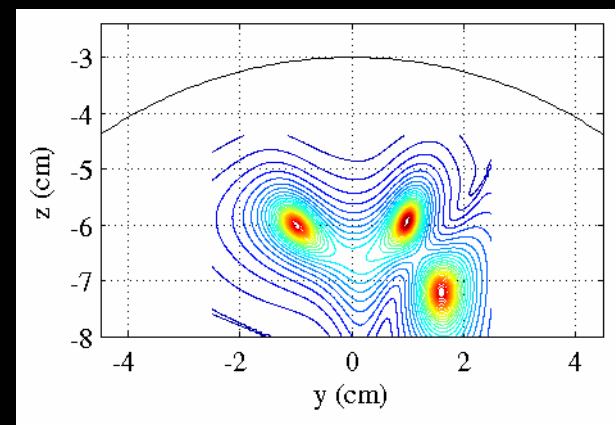
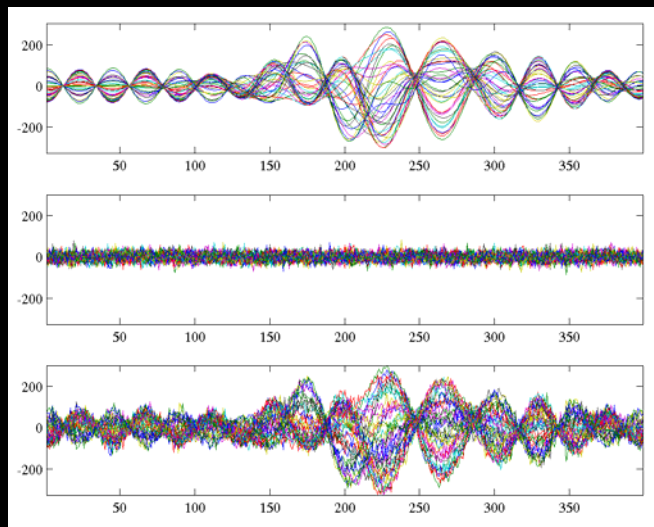
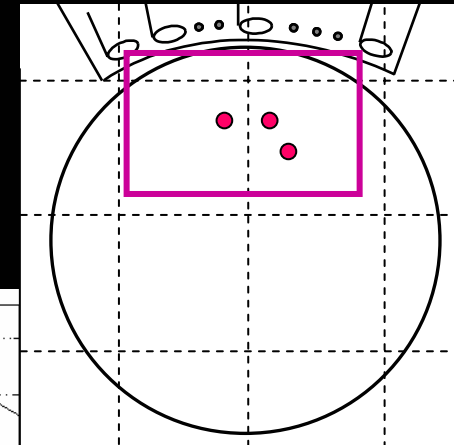
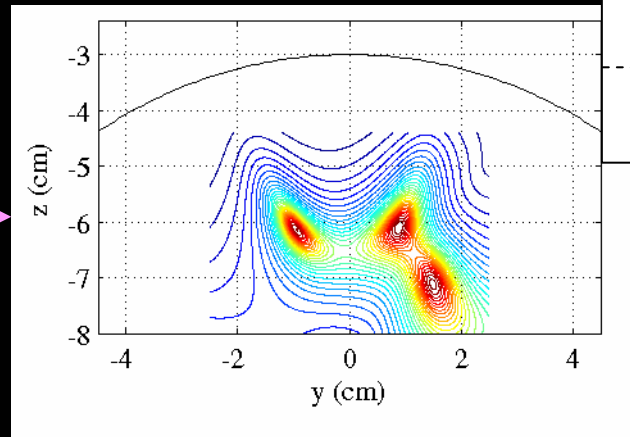
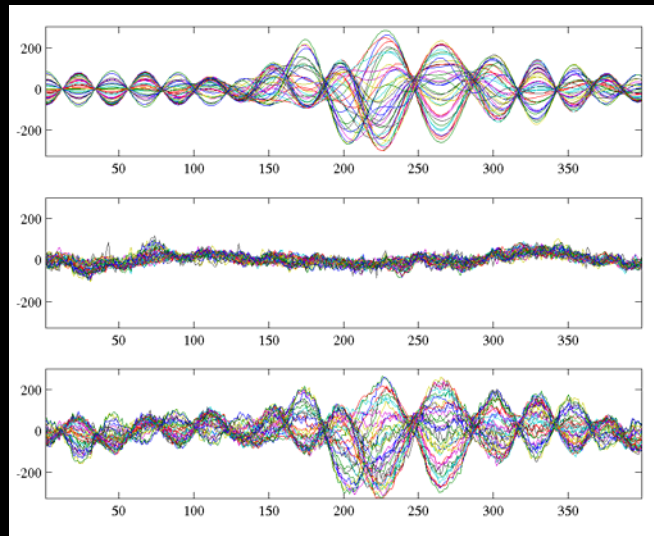


$$P_N = 0.003\sigma_2^2$$

$$P_N = 0.03\sigma_2^2$$

$P_N$  : power of noise sources,  $\sigma_2^2$  : power of the second source

# Effect of background brain activity



Reconstruction from brain-noise added data is more blurred.

# Implications of these numerical experiment results:

The background brain activity contains many incoherent activities.



Large background brain activity (brain noise) may invalidate the low-rank assumption, and may cause blurred source reconstruction

# Other causes of unsatisfactory reconstruction

## (1) Forward modeling error

Diagonal loading (Tikhonov regularization)  
(Cox. 1974)

Eigenspace projection (Project weight vector onto the signal-subspace of the measurement covariance matrix (Sekihara et al. 2002))

## (2) Sample covariance error

Beam space processing (Van Veen)

# Summary

- Reviews principles of adaptive beamformer source reconstruction.
- Discusses the equivalence between scalar and vector formulations.
- Describes two underlying assumptions:
  - 1) Sources are uncorrelated,
  - 2) Signals are low rank.
- Discusses the influences caused when these assumptions are not satisfied.
- Points out that the forward modeling error and the sample covariance error are also causes of unsatisfactory reconstruction.

**Thanks for your attention.**

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<http://www.tmit.ac.jp/~sekihara/>

The PDF version of this power-point presentation as well as PDFs of the recent publications are available.

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